The definite Integral Lecture 4

Like Pygmalion's Galatea or rabbi Loew's Golem, the integral, once created, takes on a life of its own.

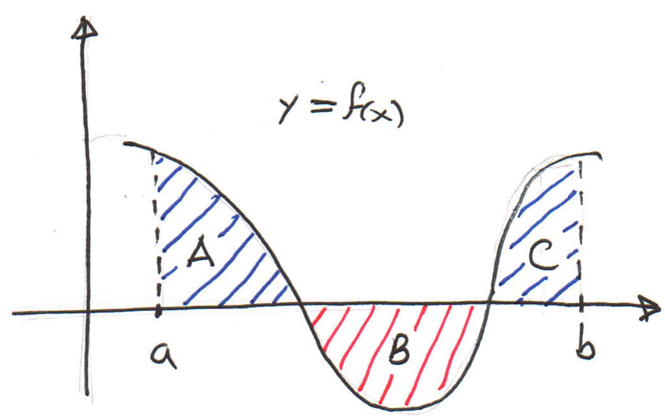
We defined $\int_a^b f(x)\,dx$ for functions $f(x) \geq 0$ over $[a, b]$. This integral represents area under $f(x)$ and above the x-axis.

$\int_a^b f(x)\,dx$ – sum up areas of thin rectangular columns which at x are $f(x)$ heigh and dx wide.

Notice that this sum is algebraically meaningful even if $f(x) < 0$. However, $f(x)\,dx$ is then negative, i.e.

$$f(x)\,dx = -|f(x)|\,dx = -\text{area of thin rectangle}.$$

In particular



If A, B, C represent areas between the x-axis and curve $y = f(x)$, then

$$\int_a^b f(x)\,dx = A - B + C.$$

or the difference of areas below $y = f(x)$ and the areas above $y = f(x)$ and below the x-axis.

The integral has the following properties:

1. $\int_a^b c\,dx = c(b-a)$ where c is any constant.

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3. $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

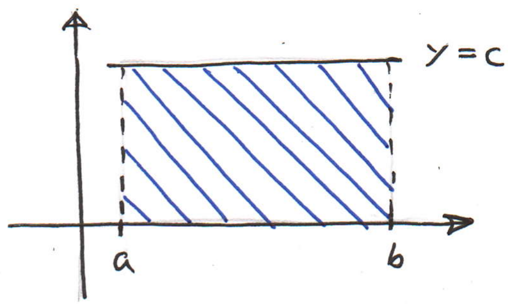
4. $\int_a^a f(x)\,dx = 0$

5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

We can see why these properties hold through insites in geometry, algebra, and limits:

1. $\int_a^b c\,dx$ represents area of rectangle.



Hence $\int_a^b c\,dx = c(b-a)$.

Alternative is to consider $\int_a^b c\,dx = \lim_{n\to\infty} R_n =$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} c\,\frac{b-a}{n} = c(b-a).$$

2. $\int_a^b [f(x)+g(x)]\,dx$ is just a sum of thin rectangular columns of height $f(x)+g(x)$ and width dx.

Since $[f(x)+g(x)]dx = f(x)\,dx + g(x)\,dx$, we can sum up areas of the form $f(x)dx$ together first and areas of the form $g(x)dx$ second.

Thus it should be clear that

$$\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Formally

$$\int_a^b \big(f(x)+g(x)\big)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \big(f(x_k)+g(x_k)\big)\Delta x_n$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} f(x_k)\Delta x_n + \sum_{k=1}^{n} g(x_k)\Delta x_n \right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_n + \lim_{n\to\infty} \sum_{k=1}^{n} g(x_k)\Delta x_n$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Same goes for $\int_a^b \big(f(x)-g(x)\big)dx$.

3. Clearly $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ as a result of the distributive property

$$c(A_1 + A_2 + \dots + A_n) = cA_1 + cA_2 + \dots + cA_n.$$

4. $\int_a^a f(x)\,dx$ is just area of rectangle of height $f(a)$ and width 0. Hence $\int_a^a f(x)\,dx = 0$.

Alternative is to apply the limit

$$\int_a^a f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a+k\frac{a-a}{n}\right)\frac{a-a}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(a)\cdot 0 = \lim_{n\to\infty} \sum_{k=1}^{n} 0 = 0.$$

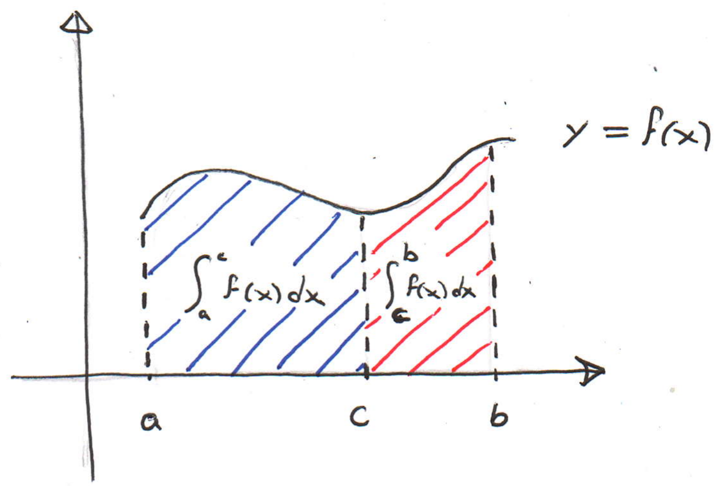
5. What happens when the bounds are interchanged? $\left(\int_a^b \to \int_b^a\ ?\right)$

$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(b+k\frac{a-b}{n}\right)\frac{a-b}{n}$$

* Formally $\int_\square^\Delta f(\cdot)\,d\cdot = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\Delta+k\frac{\Delta-\square}{n}\right)\frac{\Delta-\square}{n}$

$$= \lim_{n\to\infty} \underbrace{-\sum_{k=1}^{n} f\left(b-k\frac{b-a}{n}\right)\frac{b-a}{n}}_{L_n} = -\int_a^b f(x)\,dx.$$

6. By subdividing the interval $[a, b]$ it becomes clear that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



However it is interesting to note that c does not have to be between a and b.

If $c < a < b$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = -\int_c^a f(x)\,dx +$$

$$+ \int_c^a f(x)\,dx + \int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

If $a < b < c$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$- \int_b^c f(x)\,dx = \int_a^b f(x)\,dx$$

Ex. Evaluate the integrals the fastest way you can.

(a) $\int_0^1 \sqrt{1-x^2}\, dx$ (b) $\int_0^3 (x-1)\, dx$

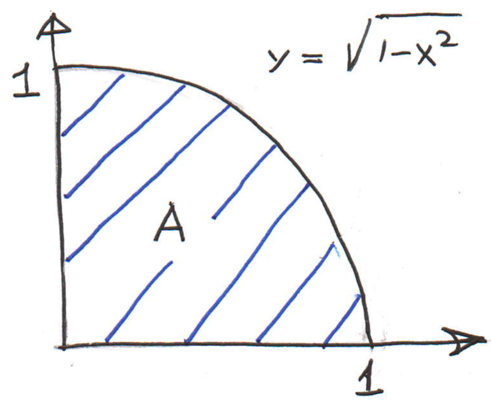
(c) $\int_{-3}^{3} \left(\sqrt{9-x^2} + x + 2\right) dx$

Solution:

(a) Set $y = \sqrt{1-x^2}$

Then $y^2 = 1 - x^2$ or $x^2 + y^2 = 1$

This is a circle!

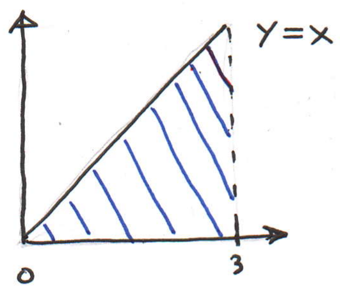


$\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{4}$ area of unit circle

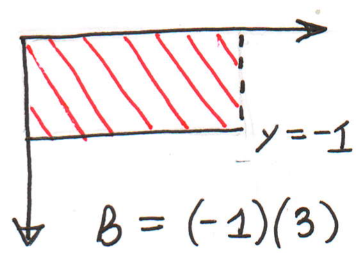
$= \frac{1}{4} \cdot \pi$

(b) $\int_0^3 x\,dx + \int_0^3 (-1)\,dx$

The first is a right triangle with height = width = 3 and the second is a rectangle of height -1 and width 3.



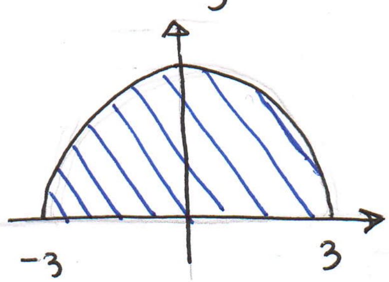
$A = \frac{9}{2}$



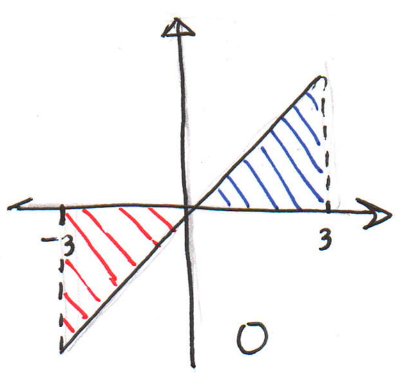
$B = (-1)(3) = -3$

Thus $\int_0^3 (x-1)\,dx = \frac{9}{2} - 3 = \frac{3}{2}$

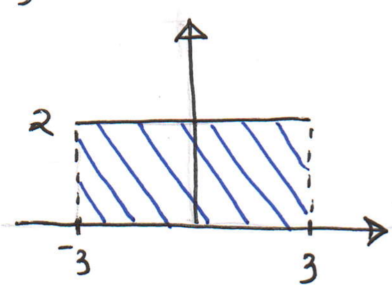
(c) $\int_{-3}^3 \sqrt{9-x^2}\,dx + \int_{-3}^3 x\,dx + \int_{-3}^3 2\,dx$



$\frac{9\pi}{2}$



0



12

Thus $\int_{-3}^3 \left(\sqrt{9-x^2} + x + \cancel{8}2\right) dx = \frac{9\pi}{2} + 12.$

Ex. If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$ find $\int_8^{10} f(x)\,dx$.

Solution:

$$\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^{8} f(x)\,dx = 17-12 = 5.$$

Comparison properties of the integral

1. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Explain why these properties are true.

Ex. Use property 3 to estimate $\int_0^1 e^{-x^2}\,dx$

Solution:

$$e^{-1^2} \leqslant e^{-x^2} \leqslant e^{-0^2} = 1$$

Hence

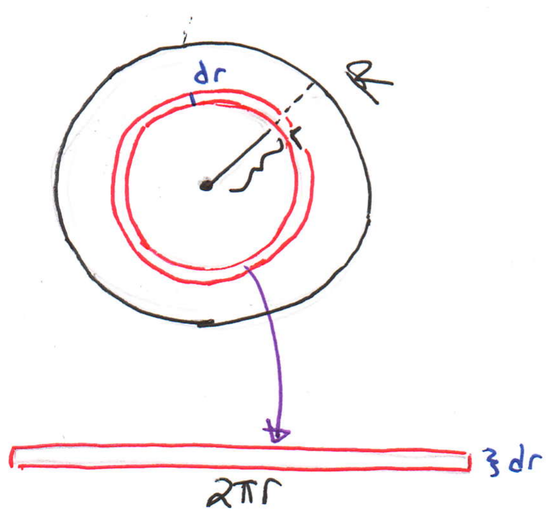
$$e^{-1}(1-0) \leqslant \int_0^1 e^{-x^2}dx \leqslant 1\cdot(1-0)$$

$$e^{-1} \leqslant \int_0^1 e^{-x^2}dx \leqslant 1.$$

The integral is a powerful tool for computing areas, volumes, and anything else that requires sums.

Ex. Find area of circle of radius R assuming circumference is $2\pi R$.

Solution:



Think of the circle as a layer of onion rings of thickness dr. The ring at a distance of r from circle's center can be unwrapped into a rectangle of length $2\pi r$ and width dr.

To find the area within the circle, it is therefore sufficient to add up all the areas of the onion rings.

Thus area of circle $= \int_0^R 2\pi r\,dr$

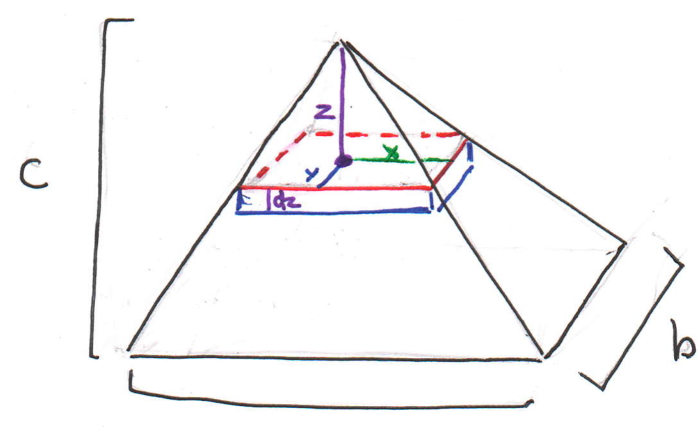
$= 2\pi \int_0^R r\,dr$

where $\int_0^R r\,dr$ is just the area of a triangle of height = width = R

$$\int_0^R r\,dr = \frac{r^2}{2}.$$

Thus $2\pi \cdot \frac{r^2}{2} = \pi r^2.$

<u>Ex.</u> Find the volume of the pyramid.



View the pyramid as a pile of thin boxes of volume $(2y)(2x)\,dz$. Now $\frac{y}{z} = \frac{\frac{1}{2}b}{c} = \frac{b}{2c}$

Similerly $\frac{x}{z} = \frac{\frac{1}{2}a}{c} = \frac{a}{2c}$

Thus $x = \frac{a}{2c} z$ and $y = \frac{b}{2c} z$

and volume of the box at z is $\left(2 \frac{a}{2c} z\right)\left(2 \frac{b}{2c} z\right) dz$

$= \frac{ab}{c^2} z^2 dz$

Volume of pyramid $\int_0^c \frac{abz^2}{c^2} dz =$

$= \frac{ab}{c^2} \int_0^c z^2 dz$

Now $\int_0^c z^2 dz = \lim_{n \to \infty} \sum_{k=1}^{n} \left(k \frac{c}{n}\right)^2 \frac{c}{n}$

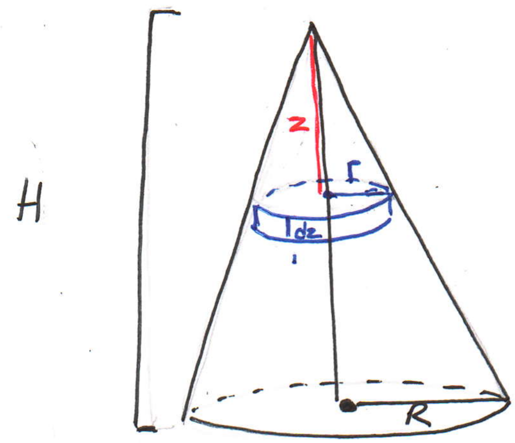
$= \lim_{n \to \infty} \left(\frac{c}{n}\right)^3 \sum_{k=1}^{n} k^2 = \lim_{R \to \infty} \left(\frac{c}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$

$= \frac{2c^3}{6} = \frac{c^3}{3}$

Hence the volume is $\frac{ab}{c^2} \cdot \frac{c^3}{3} = \frac{abc}{3}$

Ex. Find the volume of a circular cone of height H and base radius R.

Solution:



Imagine the cone to be a pile of thin cylinders. Their volume is $\pi r^2 dz$ where $\frac{r}{z} = \frac{R}{H}$. Hence $r = \frac{R}{H} z$

Thus $\pi r^2 dz = \pi z^2 \left(\frac{R}{H}\right)^2 dz = \pi \left(\frac{R}{H}\right)^2 z^2 dz$.

$$\text{Volume} = \int_0^H \pi \left(\frac{R}{H}\right)^2 z^2 dz = \pi \left(\frac{R}{H}\right)^2 \int_0^H z^2 dz$$

$$= \pi \left(\frac{R}{H}\right)^2 \frac{H^3}{3} = \frac{\pi R^2 H}{3}$$